Echelon Stock Formulation of Arborescent Distribution Systems: An Application to the Wagner-Whitin Problem

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Abstract. An arborescent distribution system is a multi-level system in which each installation receives input from a unique immediate predecessor and supplies one or more immediate successors. In this paper, it is shown that a distribution system with an arborescent structure can also be modelled using an echelon stock concept where at any instant the total echelon holding cost is accumulated at the same rate as the total conventional holding cost. The computational efficiency of the echelon model is tested on the well-known Wagner-Whitin type dynamic inventory lot-sizing problem, which is an intractable combinatorial problem from both mixed-integer programming (MIP) and constraint programming (CP) standpoints. The computational experiments show that the echelon MIP formulation is computationally very efficient compared to the conventional one, whereas the echelon CP formulation remains intractable. A CP/LP hybrid yields a substantial improvement over the pure CP approach, solving all tested instances in a reasonable time.

1 Introduction

Inventory theory provides methods for managing and controlling inventories under different policy constraints and environmental situations. A basic distribution system consists of a supply chain of stocking points arranged in levels. Customer demands occur at the first level, and each level has its stock replenished from the one above. Typically, a *holding cost* per unit of inventory is associated with each stocking point, under the assumption that a parent stocking point has a lower holding cost than any of its children. A *procurement cost* per order is also associated with each stocking point. Given customer demands for each stocking point in the first level over some planning horizon of a number of periods, the problem is then to find an optimal *policy*: a set of decisions as to when and how much to order for each stocking point, such that cost is minimised. This is a difficult combinatorial problem, to which this paper considers mixed integer programming (MIP), constraint programming (CP) and hybrid approaches.

An important consideration in the performance of all of these approaches is the *model*, i.e. the choice of decision variables and constraints used to represent the problem. For each stocking point, at each time period in the planning horizon, the *conventional* model employs one variable for the ordering decision and another for the closing inventory level. An alternative model of inventory views the distribution structure in *echelons* [8]. An echelon comprises a stocking point and all of its descendants, with an associated echelon inventory level corresponding to the combined inventories of the constituent stocking points. The *echelon holding cost* [9] captures the incremental cost of holding a unit of stock at a particular stocking point instead of at its parent.

The echelon stock formulation, with an inventory variable per echelon, has been shown to be valid for serial distribution systems [14]. This paper extends the proof of validity to *arborescent systems*, showing that the total echelon holding cost accumulates at the same rate as the total conventional holding cost.

The complexity of multi-echelon inventory problems has in the past required the use of a sequential approach to calculating the optimal policy [12,13]. Clark and Scarf [8] demonstrated that, using the echelon formulation, under certain inventory control policy and cost assumptions, the optimal policy for a serial system can be determined sequentially by first determining the optimal policy at the lowest level and then proceeding sequentially to the higher levels. This paper demonstrates empirically, via a multi-echelon version of the well-known Wagner-Whitin problem [16], that the echelon stock formulation may still improve on the conventional formulation without resorting to the sequential approach.

The paper is organised as follows. In Section 2, the concepts of arborescence and echelon stock, and in Section 3, the notation and basic definitions are given. The conventional and echelon formulations of arborescent distribution systems are presented in Section 4 and their equivalence is proved in Section 5. Section 6 illustrates the echelon MIP formulation of the Wagner-Whitin problem by a numerical example. Section 7 is devoted to numerical tests concerning the computational efficiency of echelon and conventional formulations of Wagner-Whitin problem. Conclusions are presented in the final section.

2 Multi-Echelon Systems and Echelon Stock

Figure 1 presents an illustrative multi-echelon inventory system. A multi-echelon inventory system can also be viewed as a directed network, where the nodes represent the stocking points and the linkages represent flows of goods. If the network has at most one incoming link for each node and flows are acyclic it is called an arborescence or inverted tree structure. More complex interconnected systems of facilities can exist; however, most of the work in multi-echelon inventory theory has been confined to arborescent structures [12].

Consider the following distinction between an installation stock and an echelon stock. In a serial system, the stock at installation i refers only to the stock physically at that location; however, the echelon stock at level i refers to the sum of all the stocks at installations i, $i-1$, ..., 2, 1 plus all the stock in transit between installations i, $i-1$, ..., 2, 1. This definition permits useful simplifications [8]. Under certain assumptions, a multi-stage problem can be decomposed into a set of interconnected one-stage problems, one for each echelon. In Afentakis et

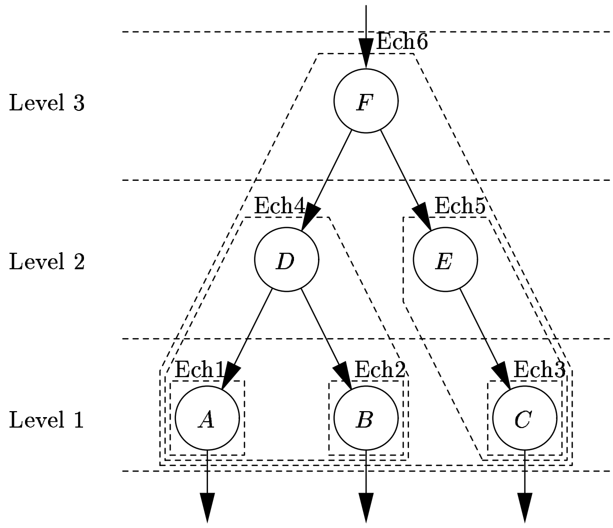

Fig. 1. An arborescent system

al. [2], Clark and Armentano [6,7], Axsater and Rosling [3], and Chen and Zheng [5], the echelon concepts are used to reformulate production/inventory systems. Silver et al. [15] give a full chapter treatment of the topic.

3 Notation and Definitions

Consider a level $m \in \{1, ..., L\}$ of an arborescent structure and let the number of stocking points in this level be N_m. For each stocking point, $W(i, m, n)$ is the set of descending $(n < m)$ or immediate ascending $(n = m + 1)$ stocking points that are in level n and connected to the ith, $i \in \{1, ..., N_m\}$, stocking point of the mth level. $G(i, m)$ is the set of all successors of stocking point i of mth level (i.e., $G(i, m) \equiv W(i, m, \{n|n < m\})$). $V(i, m)$ is the set of all stocking points that are in the first level and originate from stocking point i of the mth level (i.e., $V(i, m) \equiv W(i, m, 1)$). Each stocking point is defined by a pair of numbers (i, m), where i and m denote the stocking point and level numbers respectively.

For illustration, consider the 3-level distribution system shown in Figure 1, where $(i = 1, m = 1)$ refers to stocking point A, $(i = 2, m = 1)$ to stocking point B, $(i = 3, m = 1)$ to stocking point C, $(i = 1, m = 2)$ to stocking point D, $(i = 2, m = 2)$ to stocking point E, and $(i = 1, m = 3)$ to stocking point F:

- Set of immediate successors, $W(i, m, m - 1)$:
  $W(1, 2, 1) = \{A, B\}$, $W(2, 2, 1) = \{C\}$, $W(1, 3, 2) = \{D, E\}$
- Set of all successors, $G(i, m)$:
  $G(1, 3) = \{A, B, C, D, E\}$, $G(1, 2) = \{A, B\}$, $G(2, 2) = \{C\}$, $G(1, 1) = G(2, 1) = G(3, 1) = \emptyset$
- Immediate predecessors, $W(i, m, m + 1)$:
  $W(1, 1, 2) = D$, $W(2, 1, 2) = D$, $W(3, 1, 2) = E$, $W(1, 2, 3) = F$, $W(2, 2, 3) = F$
- Set of successors in the lowest level, $V(i, m)$:
  $V(1, 2) = \{A, B\}$, $V(2, 2) = \{C\}$, $V(1, 3) = \{A, B, C\}$

The echelon stock at stocking point i in level j at the end of period t is denoted by E_{ijt}, and e_{ij} is the echelon holding cost. The echelon holding cost at a given stocking point (i, j) is the incremental cost of holding a unit of inventory at this stocking point instead of at predecessor thereof, $W(i, j, j+1)$. The formal definitions of e_{ij} and E_{ijt} are as follows:

$$\left.\begin{aligned} e_{ij} &= c_{ij} - c_{W(i,j,j+1)} \\ E_{ijt} &= I_{ijt} + \sum_{m \in G(i,j)} I_{mt} \end{aligned}\right\} \quad t = 1, ..., T \quad j = 1, ..., L \quad i = 1, ..., N_j. \tag{1}$$

where c_{ij} is the conventional unit inventory holding cost at stocking point (i, j) and I_{ijt} is the closing inventory level at the end of period t.

4 Conventional vs Echelon Formulations

In this section, inventory holding cost expressions and balance equations for multi-echelon inventory systems are presented under both conventional and echelon holding cost charging schemes. First, the conventional scheme is addressed under the assumption that a fixed holding cost c is incurred on any unit carried in inventory over from one period to the next. Under this scheme, the conventional single period total holding cost expression can be written as in Eq.(2),

$$\sum_{j=1}^{L} \sum_{i=1}^{N_j} c_{ij} I_{ijt} \quad t = 1, ..., T. \tag{2}$$

The pertinent inventory balance equations show that the closing inventory in any stocking point in any period is the opening inventory plus the order received minus the demand met (or the amount supplied to the other stocking points of the multi-echelon system),

$$I_{ijt} = I_{ij(t-1)} + X_{ijt} - \sum_{m \in W(i,j,j-1)} X_{mt} \qquad i = 1, ..., N_j \quad t = 1, ..., T \quad j = 1, ..., L \tag{3}$$

and

$$X_{ijt}, I_{ijt} \geq 0, \tag{4}$$

where X_{ijt} is the stock replenishment amount received at stocking point (i, j) in period t. In Eq.(3), without loss of generality, delivery lead-time is taken as zero. This formulation, in which Eq.(2) is the cost expression and Eqs.(3)–(4) are the balance equations, is called *Model I* (or the conventional model).

It is shown in Section 5 that, by means of the linear transformations given in Eq.(1), *Model I* of multi-echelon systems can be rewritten as Eqs.(5)–(7),

$$\sum_{j=1}^{L} \sum_{i=1}^{N_j} e_{ij} E_{ijt} \qquad t = 1, ..., T \tag{5}$$

$$E_{ijt} = E_{ij(t-1)} + X_{ijt} - \sum_{m \in V(i,j)} d_{mt} \qquad t = 1, ..., T \quad j = 1, ..., L \quad i = 1, ..., N_j \tag{6}$$

$$0 \leq \sum_{m \in W(i,j,j-1)} E_{mt} \leq E_{ijt} \qquad t = 1, ..., T \quad j = 1, ..., L \quad i = 1, ..., N_j \tag{7}$$

where customer demands, d_{mt}, occur only at stocking points in level 1. By definition, $X_{m \in W(i,j,0)}$ and $d_{m \in V(i,j)}$ are equivalent. This alternative formulation is called *Model II* (or the echelon model). The concept behind this transformation is known in the MRP literature (for assembly systems) as "explosion" (see Afentakis and Gavish [1]).

5 Equivalence of Models *I* and *II*

The validity of echelon stock and echelon holding cost concepts in arborescent structures is now addressed and the equivalence of *Models I* and *II* is proved. To serve this purpose, we show that at any instant a policy under the echelon stock charging scheme gives the same total cost as a charging scheme based on stock physically at each installation in an arborescent structure.

Lemma 1. *Given* $E_{ijt} = I_{ijt} + \sum_{m \in G(i,j)} I_{mt}$*, the inventory balance equations of Models I and II are equivalent.*

Proof. Consider level j, $j \in \{1, ... L\}$, of *Model I.* For all stocking points in the set of all successors, $G(i,j)$, adding up the inventory balance equations of Eq.(3),

$$\sum_{m \in W(i,j,j-1)} I_{mt} = \sum_{m \in W(i,j,j-1)} I_{m(t-1)} + \sum_{m \in W(i,j,j-1)} X_{mt} - \sum_{m \in W(i,j,j-2)} X_{mt}$$

$$\sum_{m \in W(i,j,j-2)} I_{mt} = \sum_{m \in W(i,j,j-2)} I_{m(t-1)} + \sum_{m \in W(i,j,j-2)} X_{mt} - \sum_{m \in W(i,j,j-3)} X_{mt}$$

$$\vdots$$

$$\sum_{m \in W(i,j,1)} I_{mt} = \sum_{m \in W(i,j,1)} I_{m(t-1)} + \sum_{m \in W(i,j,1)} X_{mt} - \sum_{m \in W(i,j,0)} X_{mt}$$

yields

$$\sum_{m \in G(i,j)} I_{mt} = \sum_{m \in G(i,j)} I_{m(t-1)} + \sum_{m \in W(i,j,j-1)} X_{mt} - \sum_{m \in W(i,j,0)} X_{mt}. \tag{8}$$

Thereby, using the equivalence of $X_{m \in W(i,j,0)}$ and $d_{m \in V(i,j)}$, and adding Eq.(3) to Eq.(8) give the general expression,

$$\left(I_{ijt} + \sum_{m \in G(i,j)} I_{mt} \right) = \left(I_{ij(t-1)} + \sum_{m \in G(i,j)} I_{m(t-1)} \right) + X_{ijt} - \sum_{m \in V(i,j)} d_{mt}. \tag{9}$$

Making use of Eq.(1) (i.e., the definition of E_{ijt}), Eq.(9) leads to

$$E_{ijt} = E_{ij(t-1)} + X_{ijt} - \sum_{m \in V(i,j)} d_{mt} \tag{10}$$

Hence, the equivalence of Eq.(3) and Eq.(6) is shown.

Eq.(1) can be rearranged for I_{ijt} giving

$$I_{ijt} = E_{ijt} - \sum_{m \in G(i,j)} I_{mt} \tag{11}$$

and

$$I_{ijt} = E_{ijt} - \sum_{m \in W(i,j,j-1)} E_{mt}, \tag{12}$$

from which the nonnegativity constraints of I_{ijt} lead directly to Eq.(7).

In what follows, the equivalence of conventional and echelon holding cost charging schemes is addressed. The echelon holding cost, E_{ijt}, is calculated from the conventional holding cost, c_{ij}, by the rule:

$$e_{ij} = c_{ij} - c_{W(i,j,j+1)} \qquad j = 1, ..., L \quad i = 1, ..., N_j. \tag{13}$$

Schwarz and Schrage [14] show that in a serial system, at any instant

$$\sum_{j=1}^{L} \sum_{i=1}^{N_j=1} c_{ij} I_{ijt} = \sum_{j=1}^{L} \sum_{i=1}^{N_j=1} e_{ij} E_{ijt}, \qquad t = 1, ..., T$$

that is, total echelon holding cost is accumulated at the same rate as total conventional holding cost.

Here, we show that conventional and echelon charging schemes are identical not only in serial systems, but also in arborescent systems. Hence their proof is extended to cover arborescent systems.

Lemma 2. *Given* $e_{ij} = c_{ij} - \{c_m | m \in W(i,j,j+1)\}$*, the cost expressions of Models I and II are equivalent.*

Proof. Assume that an arborescent system is comprised of only a single level of stocking points. In other words, the stocking points are independent. Then $E_{i1t} = I_{i1t}$ and $e_{i1} = c_{i1}$ for $i = 1, ..., N_1$, and $\sum_i e_{i1} E_{i1t} = \sum_i c_{i1} I_{i1t}$. Now assume that a new stocking point is introduced as level 2, and all independent stocking points of level 1 are connected to this new stocking point as successors. Then it follows

$$\begin{aligned} e_{12} &= c_{12} \\ e_{i1} &\leftarrow e_{i1} - c_{12} \qquad i = 1, ..., N_1, \\ E_{12t} &= I_{12t} + \sum_{i=1}^{N_1} E_{i1t} \end{aligned}$$

from which the conventional inventory cost increase may be calculated as

$$e_{12} E_{12t} - \sum_{i=1}^{N_1} c_{12} E_{i1t} = c_{12} \left(E_{12t} - \sum_{i=1}^{N_1} E_{i1t} \right) = c_{12} I_{12t}$$

Since this is the actual amount of increase in the conventional inventory cost, one can conclude that the echelon cost charging scheme gives the correct conventional inventory cost for a 2-echelon system.

Now assume an inventory system of N_{j-1} independent $j-1$ level systems. A new level, level j, is introduced and all $j-1$ level systems are connected to it as successors. Such a restructuring does not affect the echelons of the lowest $j-2$ levels. However, the following modifications take place:

$$\begin{aligned} &e_{1j} = c_{1j} \\ &e_{i(j-1)} \leftarrow e_{i(j-1)} - c_{1j} \qquad i = 1, ..., N_{j-1}, \\ &E_{1jt} = I_{1jt} + \sum_{i=1}^{N_{j-1}} E_{i(j-1)t}. \end{aligned}$$

Thus, the echelon holding cost expression increases by

$$e_{1j}E_{1jt} - \sum_{i=1}^{N_{j-1}} c_{1j}E_{i(j-1)t} = c_{1j}\left(E_{1jt} - \sum_{i=1}^{N_{j-1}} E_{i(j-1)t}\right) = c_{1j}I_{1jt}.$$

Since the actual amount of increase in the echelon cost is equal to the amount of increase in the conventional inventory cost for an arborescent structure with j levels, this finishes the induction step, which completes the proof.

Theorem 1. *An alternative formulation of arborescent distribution systems follows from the echelon stock and echelon holding cost definitions.*

Proof. From Lemmas (1) and (2).

6 An Illustrative Example: Wagner-Whitin Problem

The Wagner-Whitin problem [16] describes the single stocking point planning of ordering and stocking a certain product over a discrete time planning horizon. The deterministic demand for all periods is to be satisfied, and the total sum of fixed procurement and linear holding costs is to be minimised. In this section a multi-echelon version of the Wagner-Whitin type dynamic inventory lot-sizing problem (problem 040 at www.csplib.org) is formulated using both conventional and echelon approaches. To serve this purpose, the multi-echelon structure given in Figure 1 is used assuming the period demands presented in Table 1.

Table 1. Period demands

Installation	P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
A	100	100	100	100	100	100	100	100	100	100
B	50	200	50	50	200	250	250	100	150	150
C	250	50	350	50	250	50	250	50	350	50

Table 2. Problem parameters and optimal replenishment amounts, X_{ijt}

Installation /Echelon	Holding Cost, c	Echelon Cost, e	Procurement Cost, c_o	Optimal replenishment amounts, X_{ijt} P1	P2	P3	P4	P5	P6	P7	P8	P9	P10
A	10	4	1,000	200	–	200	–	200	–	200	–	200	–
B	30	24	3,000	50	300	–	–	200	250	350	–	150	150
C	20	14	1,000	300	–	400	–	300	–	300	–	400	–
D	6	1	5,000	750	–	–	–	1700	–	–	–	–	–
E	6	1	7,000	700	–	–	–	1000	–	–	–	–	–
F	5	5	10,000	1450	–	–	–	2700	–	–	–	–	–

The initial inventory level is taken as zero and the replenishment lead-time is set to zero in all stocking points. A fixed procurement cost, c_o, is incurred when a replenishment order is placed, irrespective of order size. Table 2 gives the other parameters of the problem. Both conventional and echelon type formulations for the multi-echelon Wagner-Whitin problem are presented below. In these models, M denotes a large number and δ_{ijt} is a binary decision variable that takes the value of 1 if a replenishment order is placed in period t and 0 otherwise.

Conventional Model:

min

$$\sum_{t=1}^{T}\sum_{j=1}^{L}\sum_{i=1}^{N_j}\left(c_{ij}I_{ijt} + c_{o_{ij}}\delta_{ijt}\right)$$

subject to

$(i = 1, \ldots, N_j \quad j = 1, \ldots, L \quad t = 1, \ldots, T)$

$$I_{ijt} = I_{ij(t-1)} + X_{ijt} - \sum_{m \in W(i,j,j-1)} X_{mt}$$

$$X_{ijt} \le M\delta_{ijt}$$

$$X_{ijt}, I_{ijt} \ge 0, \quad \delta_{ijt} \in \{0,1\}$$

Echelon Model:

min

$$\sum_{t=1}^{T}\sum_{j=1}^{L}\sum_{i=1}^{N_j}\left(e_{ij}E_{ijt} + c_{o_{ij}}\delta_{ijt}\right)$$

subject to

$(i = 1, \ldots, N_j \quad j = 1, \ldots, L \quad t = 1, \ldots, T)$

$$E_{ijt} = E_{ij(t-1)} + X_{ijt} - \sum_{m \in V(i,j)} d_{mt}$$

$$X_{ijt} \le M\delta_{ijt}$$

$$X_{ijt} \ge 0, \quad \delta_{ijt} \in \{0,1\}$$

$$E_{ijt} \ge \sum_{m \in W(i,j,j-1)} E_{mt} \ge 0$$

The optimal inventory replenishment policy, with a total cost of 135,700 units, is presented in Table 2. The difference between two models is significant. In the conventional case, the search tree includes 531 nodes, whereas the corresponding search tree for the echelon model has only 213 nodes.

7 Computational Experiments

To get a better indication of the difference between the two models, in this section computational tests are performed on a wider set of problems, using different solution techniques. The tests are performed on a 1.2GHz Pentium-3 machine using mathematical programme solvers Xpress-MP 2003B [10] and Ilog Cplex 8.1 and constraint solver Ilog Solver 5.3. Xpress-MP and Cplex are both used with their default settings. The models are tested on problems generated for 6-stocking point multi-level systems with three different structures, namely: arborescent systems; serial systems, and warehouse-retailer systems.

In the test problems, the structure given in Figure 1 is used as a design of arborescent systems. Serial systems are represented by a 6-level structure, in which A is the lowest level stocking point where the external demand is met and F is the highest level (level 6) stocking point where the external supply is received. In a warehouse-retailer system, a single second level warehouse supplies a number of first level retailers. The computational tests are performed on a 1-warehouse (denoted by F) 5-retailer (denoted by A to E) structure. The pertinent demand data for arborescent, serial and warehouse-retailer systems are presented in Table 3.

Table 3. Period demands (I: Arborescent; II: Serial; III: Warehouse-Retailer)

		P1	P2	P3	P4	P5	P6	P7	P8	P9	P10	P11	P12	P13	P14	P15	P16	P17	P18
	A	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
I.	*B*	50	200	50	50	200	250	250	100	150	150	50	200	50	200	50	50	200	250
	C	250	50	350	50	250	50	250	50	350	100	50	50	250	50	350	50	250	50
II.	*A*	50	200	250	100	150	250	50	100	150	150	100	200	50	200	250	100	150	250
	A	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100	100
	B	50	200	50	50	200	250	250	100	150	150	50	200	50	50	200	250	250	100
III.	*C*	250	50	350	50	250	50	250	50	350	50	250	50	350	50	250	50	250	50
	D	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50	50
	E	50	100	150	200	250	250	200	150	100	50	50	100	150	200	250	250	200	150

Three different planning horizon lengths –10, 12 and 18 periods– are used in the experiments. The number of test problems, generated for different structures and costs, totals to 35. The instances are given in Table 4.

7.1 Implied Constraints

During numerical experiments a number of implied constraints (IC) are incorporated into the conventional and echelon models to enhance performance. These constraints are detailed below:

IC1 All stocking points must have zero inventory at the end of the last period in an optimal solution. Hence, the corresponding inventory variables are preset.

IC2 In an optimal solution, if a parent node places an order, at least one of its children must also. Consider that, if no children make an order, the parent node incurs a holding cost that can be removed by delaying the order until a subsequent period when at least one child does place an order.

IC3 An upper bound can be derived for the inventory variables in the conventional formulation by considering that it is only worth holding stock at a node if it is cheaper than ordering it at the next period. That is: $I_{ijt}c_{ij} \leq c_{o_{ij}} + I_{ijt}c_m$, $m \in W(i, j, j+1)$, which simplifies to: $I_{ijt} \leq \frac{c_{o_{ij}}}{c_{ij} - c_m}$. This can easily be applied to the echelon model by substituting in the equality: $I_{ijt} = E_{ijt} - \sum_{m \in G(i,j)} I_{mt}$.

IC4 Similarly, an upper bound can be derived for the order variables at the leaf nodes. The principle is the same: it is only worth ordering stock not absorbed by demand at the current period if it is cheaper than waiting and ordering in a subsequent period. Consider first a bound based on deferring an order

Table 4. Test Problems: Arborescent (A), Serial (S) and Warehouse-Retailer (W)

Case	Holding Costs c	Echelon Costs e	Procurement Costs c_o
	[A, B, C,D,E,F]	[A,B,C,D,E,F]	[A, B, C, D, E, F]
A-1	[3, 3, 3, 2, 2, 1]	[1, 1, 1, 1, 1, 1]	[1000,1000,1000,1000,1000,1000]
A-2	[4, 4, 4, 2, 2, 1]	[2, 2, 2, 1, 1, 1]	[1000,1000,1000,1000,1000,1000]
A-3	[4, 4, 4, 3, 3, 2]	[1, 1, 1, 1, 1, 2]	[1000,1000,1000,1000,1000,1000]
A-4	[3, 4, 5, 2, 4, 1]	[1, 2, 1, 1, 3, 1]	[1000,1000,1000,1000,1000,1000]
A-5	[3, 6, 3, 2, 2, 1]	[1, 4, 1, 1, 1, 1]	[1000,1000,1000,1000,1000,1000]
A-6	[6, 5, 4, 3, 2, 1]	[3, 2, 2, 2, 1, 1]	[1000,1000,1000,1000,1000,1000]
A-7	[3, 4, 6, 2, 5, 1]	[1, 2, 1, 1, 4, 1]	[1000,1000,1000,1000,1000,1000]
A-8	[5, 6, 3, 4, 2, 1]	[1, 2, 1, 3, 1, 1]	[1000,1000,1000,1000,1000,1000]
A-9	[4, 6, 5, 2, 3, 1]	[2, 4, 2, 1, 2, 1]	[1000,1000,1000,1000,1000,1000]
A-10	[7, 6, 5, 4, 3, 2]	[3, 2, 2, 2, 1, 2]	[1000,1000,1000,1000,1000,1000]
S-1	[6, 5, 4, 3, 2, 1]	[1, 1, 1, 1, 1, 1]	[1000,1000,1000,1000,1000,1000]
S-2	[6, 5, 4, 3, 2, 1]	[1, 1, 1, 1, 1, 1]	[1000,1500,2000,2500,3000,3500]
S-3	[6, 5, 4, 3, 2, 1]	[1, 1, 1, 1, 1, 1]	[500, 500, 500, 500, 500, 500]
S-4	[12,10, 8, 6, 4, 2]	[2, 2, 2, 2, 2, 2]	[1000,1000,1000,1000,1000,1000]
S-5	[12,10, 8, 6, 4, 2]	[2, 2, 2, 2, 2, 2]	[1000,1500,2000,2500,3000,3500]
S-6	[12,10, 8, 6, 4, 2]	[2, 2, 2, 2, 2, 2]	[2000,2000,2000,2000,2000,2000]
S-7	[18,15,12, 9, 6, 3]	[3, 3, 3, 3, 3, 3]	[1000,1000,1000,1000,1000,1000]
S-8	[18,15,12, 9, 6, 3]	[3, 3, 3, 3, 3, 3]	[1000,1500,2000,2500,3000,3500]
S-9	[18,15,12, 9, 6, 3]	[3, 3, 3, 3, 3, 3]	[2000,2000,2000,2000,2000,2000]
S-10	[30,10, 5, 3, 2, 1]	[20, 5, 2, 1, 1, 1]	[1000,1000,1000,1000,1000,1000]
S-11	[30,10, 5, 3, 2, 1]	[20, 5, 2, 1, 1, 1]	[1000,1500,2000,2500,3000,3500]
S-12	[30,10, 5, 3, 2, 1]	[20, 5, 2, 1, 1, 1]	[2000,2000,2000,2000,2000,2000]
S-13	[32,16, 8, 4, 2, 1]	[16, 8, 4, 2, 1, 1]	[1000,1000,1000,1000,1000,1000]
S-14	[32,16, 8, 4, 2, 1]	[16, 8, 4, 2, 1, 1]	[1000,1500,2000,2500,3000,3500]
S-15	[32,16, 8, 4, 2, 1]	[16, 8, 4, 2, 1, 1]	[2000,2000,2000,2000,2000,2000]
W-1	[2, 2, 2, 2, 2, 1]	[1, 1, 1, 1, 1, 1]	[1000,1000,1000,1000,1000,1000]
W-2	[3, 3, 3, 3, 3, 1]	[2, 2, 2, 2, 2, 1]	[1000,1000,1000,1000,1000,1000]
W-3	[4, 4, 4, 4, 4, 1]	[3, 3, 3, 3, 3, 1]	[1000,1000,1000,1000,1000,1000]
W-4	[2, 4, 2, 4, 2, 1]	[1, 3, 1, 3, 1, 1]	[1000,1000,1000,1000,1000,1000]
W-5	[3, 6, 3, 6, 3, 1]	[2, 5, 2, 5, 2, 1]	[1000,1000,1000,1000,1000,1000]
W-6	[2, 3, 4, 3, 2, 1]	[1, 2, 3, 2, 1, 1]	[1000,1000,1000,1000,1000,1000]
W-7	[4, 3, 2, 3, 4, 1]	[3, 2, 1, 2, 3, 1]	[1000,1000,1000,1000,1000,1000]
W-8	[6, 5, 4, 3, 2, 1]	[5, 4, 3, 2, 1, 1]	[1000,1000,1000,1000,1000,1000]
W-9	[2, 3, 4, 5, 6, 1]	[1, 2, 3, 4, 5, 1]	[1000,1000,1000,1000,1000,1000]
W-10	[3, 2, 4, 2, 3, 1]	[2, 1, 3, 1, 2, 1]	[1000,1000,1000,1000,1000,1000]

into the next period: $(X_{i1t} - d_{i1t})c_{i1} \leq c_{o_{i1}} + c_m(X_{i1t} - d_{i1t})$ which can be re-arranged: $X_{i1t} \leq d_{i1t} + \frac{c_{o_{i1}}}{c_{i1} - c_m}$ This can be generalised to consider deferring an order into any of the following periods up to the planning horizon: $X_{i1t} \leq \min_{t'=t..T} \sum_{i=t}^{t'} d_{i1t} + \frac{c_{o_{i1}}}{(t'-t+1)(c_{i1}-c_m)}$

7.2 MIP

The solution times obtained to test problems using Xpress–MP and Cplex under conventional and echelon formulations, with and without implied constraints, are given in Table 5. The results indicate that, irrespective of problem structure, costs involved, or MIP solver used, for the Wagner-Whitin problem the echelon formulation of a multi-echelon system is more tractable than the conventional one. The only exception to this remark is the warehouse-retailer type distribution systems which are solved by using Cplex. In these problems the computational performance of conventional and echelon models are very close and mixed.

Table 5. Solution Times (in secs): Arborescent (A), Serial (S) and Warehouse-Retailer (W) cases
"-" indicates no solution within 1 hour, (.) shows the total number of nodes visited

Case	Xpress–MP A,S: 12 periods W: 10 periods						Cplex A,S,W: 18 periods					
	No IC		IC1,2		IC1–4		No IC		IC 1,2		IC 1–4	
	Conventional	Echelon	Conventional	Echelon	Conventional	Echelon	Conventional	Echelon	Conventional	Echelon	Conventional	Echelon
A1	440 (399851)	3.5 (36507)	80.3 (97674)	1.4 (1436)	861 (89296)	0.9 (886)	1279 (493797)	227 (74830)	102 (34126)	29.8 (8336)	54.4 (14428)	3.7 (6203)
A2	–	8.4 (42523)	884 (595653)	9.3 (11034)	162 (186808)	7.1 (7280)	1260 (671756)	171 (67390)	114 (54112)	3.2 (13288)	31.5 (12631)	15.3 (5399)
A3	690 (477737)	18.9 (21163)	96.8 (123635)	3 (4106)	361 (78354)	4.2 (5359)	97.8 (33181)	274 (28881)	37.7 (9838)	18.5 (5812)	12.7 (3143)	9.38 (2356)
A4	–	11.7 (21011)	24.4 (36680)	2.5 (3480)	251 (67659)	2.7 (3478)	281 (132426)	8.0 (36922)	45.9 (16092)	24.5 (9084)	18.9 (5535)	18.0 (6823)
A5	–	17.9 (27599)	855 (531922)	1.8 (2102)	77.6 (104535)	1.7 (2060)	959 (379492)	360 (159020)	94.0 (36533)	44.1 (17678)	46.8 (16652)	3.2 (12996)
A6	–	879 (111988)	642 (464545)	3 (4690)	41.7 (61536)	3.9 (4414)	537 (275143)	50.2 (20758)	74.2 (35293)	17.3 (8227)	24.4 (11658)	9.38 (3077)
A7	115 (139905)	5.5 (8846)	14.5 (23846)	4.6 (5997)	76.2 (89690)	2.1 (2507)	134 (56468)	363 (22447)	23.2 (7816)	19.7 (6857)	14.9 (5219)	18.5 (5820)
A8	–	51 (58541)	380 (360470)	10.3 (12037)	111 (106018)	0.1 (28571)	502 (224897)	107 (44392)	66.6 (23565)	2.1 (7734)	45.4 (13932)	21.3 (7703)
A9	–	11.2 (19275)	169 (208679)	2.9 (4285)	3.4 (49448)	1.7 (2287)	158 (8732)	57.6 (26849)	47.6 (22648)	16.9 (8169)	9.53 (3867)	9.27 (3080)
A10	1068 (702604)	2.1 (3741)	3.2 (4871)	0.5 (473)	3.8 (5515)	0.8 (829)	19.1 (9151)	9.15 (4795)	12.0 (4686)	4.89 (1922)	3.02 (892)	1.86 (589)
S1	858 (596027)	20.9 (26425)	3.9 (6244)	0.3 (203)	3.3 (5141)	0.2 (90)	–	175 (44185)	0.4 (18682)	3.53 (745)	56.3 (25594)	3.25 (466)
S2	363 (338530)	6.1 (8946)	1 (1105)	0.3 (147)	0.9 (988)	0.3 (134)	–	692 (389667)	0.1 (12997)	3.40 (631)	27.6 (14384)	2.87 (491)
S3	–	20.7 (30696)	7.4 (14209)	0.3 (202)	2.8 (3607)	0.2 (51)	–	15.2 (5302)	9.31 (8331)	1.53 (156)	9.24 (5154)	1.59 (163)
S4	–	6.7 (12845)	7.8 (14598)	0.3 (200)	2.7 (3623)	0.2 (51)	–	15.0 (5302)	9.27 (8331)	1.52 (156)	9.24 (5154)	1.61 (163)
S5	–	10.2 (17188)	2.3 (2686)	0.3 (185)	1.7 (2332)	0.5 (215)	–	316 (143704)	94.5 (70338)	7.58 (1510)	355 (33206)	4.15 (689)
S6	902 (608622)	45 (53773)	4.2 (6648)	0.3 (201)	3.2 (5028)	0.2 (90)	–	175 (44169)	29.8 (18682)	3.53 (745)	57.6 (26140)	3.24 (466)
S7	103 (186578)	3.6 (5391)	0.5 (754)	0.2 (119)	1.9 (2538)	0.3 (219)	357 (457990)	6.11 (2954)	6.72 (6135)	1.20 (198)	4.11 (1948)	1.11 (183)
S8	–	10.8 (17863)	4.9 (9015)	0.6 (569)	3.7 (5477)	0.4 (249)	–	412 (171423)	95.0 (80686)	5.21 (1161)	69.4 (53278)	5.84 (1099)
S9	–	27.7 (39264)	4.6 (7248)	0.3 (192)	21.3 (30991)	0.4 (257)	–	120 (63577)	47.8 (36047)	2.31 (454)	3.8 (20987)	2.78 (589)
S10	2800 (1163936)	11.4 (16459)	7 (15330)	0.3 (332)	9.7 (13742)	0.2 (191)	–	60.7 (36554)	33.4 (32295)	4.67 (1597)	16.5 (12169)	2.82 (940)
S11	407 (353182)	13.4 (16661)	0.9 (1449)	0.2 (95)	1.1 (1673)	0.3 (109)	–	179 (112559)	16.8 (17219)	2.89 (989)	13.2 (12017)	2.29 (747)
S12	1540 (796290)	31.3 (37976)	4 (7776)	0.5 (328)	10 (16943)	0.5 (147)	–	366 (226569)	41.3 (41326)	5.74 (2013)	36.7 (32564)	5.16 (1509)
S13	270 (296085)	1.8 (2328)	5.7 (10192)	0.2 (130)	7 (10932)	0.4 (334)	1460 (2612382)	31.7 (20213)	25.7 (32875)	2.13 (747)	14.5 (15854)	1.43 (535)
S14	77.3 (123646)	2 (2110)	2.1 (4673)	0.2 (110)	2.2 (3296)	0.2 (109)	–	8.1 (60875)	40.9 (45025)	2.95 (924)	8.0 (25268)	2.27 (830)
S15	–	6.7 (6620)	4.7 (8789)	0.9 (950)	10 (16053)	0.5 (347)	–	323 (161240)	58.9 (60760)	5.63 (1592)	48.3 (48538)	3.47 (1144)
W1	–	110 (148698)	3.1 (69549)	6.3 (14612)	29 (58612)	3 (6848)	210 (129114)	352 (234222)	71.2 (45407)	80.8 (46166)	55.8 (33547)	54.3 (33837)
W2	–	19.3 (44651)	2.9 (6198)	1.4 (3685)	1.8 (3031)	1.1 (2448)	152 (103122)	193 (126742)	35.1 (27101)	353 (38514)	29.0 (20477)	37.5 (29287)
W3	164 (201504)	4.6 (6323)	0.3 (353)	0.3 (565)	0.8 (1181)	0.2 (358)	54.1 (42531)	271 (58695)	38.7 (30769)	3.7 (25521)	12.7 (10217)	18.7 (14362)
W4	–	209 (248174)	226 (278262)	18.7 (41808)	2.1 (47635)	8.9 (21369)	138 (74797)	139 (79696)	93.4 (66715)	54.6 (30338)	25.4 (12696)	41.4 (27902)
W5	–	8.6 (18515)	4 (7809)	1.8 (4490)	1.5 (2385)	1.8 (3689)	220 (171503)	383 (278423)	92.4 (84474)	259 (44689)	36.2 (30352)	50.7 (54070)
W6	1800 (336564)	81.8 (135496)	2.9 (5854)	2.9 (7057)	6.8 (13850)	2.8 (6902)	96 (63180)	158 (95023)	37.9 (24597)	46.2 (28348)	3.1 (23362)	34.5 (22559)
W7	–	7.6 (16921)	14.2 (30084)	0.9 (1708)	3.5 (7025)	0.9 (1379)	66.0 (43048)	76.3 (46087)	38.0 (21144)	39.4 (25396)	25.4 (13966)	18.4 (11209)
W8	87.8 (153269)	2.3 (4113)	0.6 (910)	0.4 (650)	0.3 (263)	0.3 (351)	71.1 (53948)	57.2 (36544)	0.4 (24289)	30.4 (22446)	13.6 (8801)	14.2 (10844)
W9	60.1 (112097)	5 (38850)	3.2 (6829)	5.3 (12203)	1.6 (3266)	3.4 (7755)	100 (69218)	64.6 (42328)	34.7 (27252)	56.7 (42253)	25.0 (20828)	8.7 (25977)
W10	1890 (940787)	6.3 (14956)	3.6 (7787)	1.8 (4172)	0.7 (1160)	0.7 (1420)	270 (211021)	204 (157379)	77.2 (53866)	55.4 (40103)	255 (40274)	54.1 (44770)

In almost half of the cases (30 out of 70) examined, the conventional model without ICs cannot produce the optimal solution in 1 hour. However, using the echelon formulation we were able to determine the optimal solution within an hour in all cases. The maximum solution time required by Xpress–MP (for 10 and 12 period problems) is less than 4 minutes, whereas this value is almost 11 minutes for Cplex (for 18 period problems).

The introduction of implied constraints IC1 and 2 dramatically improves the computational performance of both models. The improvement is especially significant in the serial systems where IC2, $\delta_{m \in W(i,j,j+1)} \leq \delta_{ijt}$ is very strong. However, when the design of distribution system exhibits a warehouse-retailer characteristic with many successors, the IC gets weaker, $\delta_{ijt} \leq \sum_{m \in W(i,j,j-1)} \delta_{mt}$. A further substantial improvement results from adding IC3 and 4. However, these constraints are observed to be weaker compared to IC1 and 2. In certain instances, it is even observed that IC3 and 4 have adverse effects on the computational performance.

The above observations clearly show that although in the Wagner-Whitin type problem an arborescent system cannot be interpreted as a nested set of echelons, the echelon formulation is still favoured to the conventional formulation. It should also be noted that the implied constraints play a significant role in the computational performance.

7.3 CP and CP/LP Hybrid

Initial experimentation with a pure constraint satisfaction model yielded disappointing results, with the solver unable to solve the arborescent problems in a reasonable time. Therefore, experiments were also performed with a hybrid CP/LP solver using Ilog Hybrid 1.3.1 to combine Solver and Cplex. The models used for the hybrid are essentially the same as the MIP models presented in Section 6. It is possible to remove the delta variables from both the conventional and echelon models by reifying the constraint $(X_{ijt} > 0)$. This reification can be used in the echelon holding cost expressions as follows (and similarly for the conventional model): $\sum_{t=1}^{T} \sum_{j=1}^{L} \sum_{i=1}^{N_j} \left(e_{ij} E_{ijt} + c_{o_{ij}} (X_{ijt} > 0) \right)$. However, the delta variables are a crucial part of the linear program. Preliminary experiments show that the hybrid performs very poorly without them. Therefore the delta variables are kept in the hybrid model, but the following constraints are added: $d_{ijt} = (X_{ijt} > 0)$. By inspection, this is stronger than the big-M inequality and can be used by the constraint solver component of the hybrid for propagation.

Implied Constraints Both formulations are enhanced through the addition of implied constraints IC1-4. Two further non-linear implied constraints are also added. The first ($IC5$) exploits the fact that, in an optimal solution, an order is only made at a node when the inventory is 0. This principle is also known as the zero-inventory ordering policy of Wagner-Whitin solution. This can be seen by considering that if an order is made at a node with some stock at period t, the cost incurred by holding that stock from period $t-1$ to t can be removed by increasing the size of the order at period t.

The second ($IC6$) reduces the domains of the X (order) variables by exploiting the fact that, in an optimal solution, the sizes of all orders made are composed from the demands of the children of the associated node for a continuous stretch of time from between the current period to the end of the planning horizon [17]. It is therefore possible to enumerate the domain elements for each X variable, replacing the simple upper/lower bounds representation. The time complexity of this process is exponential in the number of leaves beneath the order node in question, but can usefully be applied when the number of leaves is small.

Results Experiments were performed on the same problems as those used with Xpress–MP (see Tables 4 and 5). The results are presented in Table 6. Due to the branching factor, it is not feasible to use $IC6$ on the warehouse problems. The delta variables were used to branch on after preliminary experimentation showed that this was more effective than using the X variables. The branching variable was selected by preferring variables with a fractional value (in the solution to the linear relaxation) as close to 0.5 as possible [11]. Having selected a variable, branching was performed using four heuristics. The first ($H1$) always tries 1 before 0 in order to encourage propagation based on a decision having been made to place an order. Heuristics $H2$ and $H3$ branch on the smallest and largest variation in pseudo-cost [4] respectively. Finally, heuristic ($H4$) branches on the value farthest away from that assigned by the relaxation. Cuts are added via Cplex at the root node.

Compared with the MIP solvers, it is immediately clear that the hybrid takes longer to solve the instances tested. One of the chief reasons for this is that the pure MIP solvers can search many more nodes per second (around five times) than the hybrid. Particularly when domain reduction ($IC6$) is used, the search tree when using the conventional model is often smaller than that of Xpress–MP. The fact that this is not reflected in the time taken means that the reduction in search is not sufficient to compensate for the overhead of maintaining the hybrid.

The echelon model does not provide the clear advantage for the hybrid that it does for the MIP models. For the serial problems, the effect is largely positive, with a smaller search tree often explored when using the echelon model. For the arborescent problems, the echelon model is able to exploit domain reduction better than the conventional model, resulting in the echelon model performing better for a greater proportion of the instances. Performance of the echelon model versus the conventional model on the warehouse problems is dependent on the heuristic used. Sometimes an echelon model results in a smaller search tree, but longer time taken. This suggests that the use of the echelon model incurs some overhead.

Since, as confirmed in the MIP experiements, the echelon model provides a tighter relaxation, the occasions where the echelon model inhibits performance are probably due to the effects of constraint propagation. Specifically, constraint propagation will act to assign variables in the linear program, thus influencing the branching heuristic. This suggests that not only the model but the branching heuristic should be a hybrid, considering both the linear program and the constraint program.

8 Conclusion

This paper has extended Schwarz and Schrage's [14] proof of the validity of the echelon formulation for serial distribution systems to arborescent systems. The utility of this formulation in a MIP setting was confirmed in an empirical analysis using the well known Wagner-Whitin problem. The success of the echelon formulation was less clear-cut in conjunction with the hybrid CP/LP solver. This was ascribed to the influence of constraint propagation on the branching heuristic, which considers the linear relaxation only. An important piece of future work is to develop a branching heuristic that considers both the linear program and the constraint program. Further future work will consider the echelon formulation in other arborescent structures and in different operating environments, such as backlogging of unsatisfied demand at the lowest echelon

References

1. Afentakis, P., Gavish, B. Bill of material processor algorithms -time and storage complexity analysis. Technical report, Graduate School of Management, University of Rochester (1983)
2. Afentakis, P., Gavish, B., Karmarkar, U. Computationally efficient optimal solutions to the lot-sizing problem in assembly systems. Management Science **30** (1984) 222–239.
3. Axsater, S., Rosling, K. Notes: Installation vs. echelon stock policies for multilevel inventory control. Management Science **39** (1993) 1274–1279.
4. Benichou, M., Gauthier, J. M., Hentges, G., Ribiere, G. Experiments in mixed integer linear programming. Mathematical Programming **1** (1971) 76-94.
5. Chen, F., Zheng, Y.S. One-warehouse multiretailer systems with centralized stock information. Operations Research **45** (1997) 275–287.
6. Clark, A.R., Armentano, V.A. Echelon stock formulations for multi-stage lot-sizing with lead-times. International Journal of Systems Science **24** (1993) 1759–1775.
7. Clark, A.R., Armentano, V.A. A heuristic for a resource-capacitated multi-stage lot-sizing problem with lead times. Journal of the Operational Research Society **46** (1995) 1208–1222.
8. Clark, A.J., Scarf, H. Optimal policies for a multi-echelon inventory problem. Management Science **6** (1960) 475–490.
9. Crowston, W.B., Wagner, M.H., Williams, J.F. Economic lot size determination in multi-stage assembly systems. Management Science **19** (1973) 517–527.
10. Optimization, D. Xpress-MP Essentials: An Introduction to Modeling and Optimization. Dash Associates, New Jersey (2002)
11. Padberg, M.W., Rinaldi, G. A branch-and-bound algorithms for the resolution of large-scale symmetric traveling salesman problems. SIAM Review, **33**(1) (1991) 60–100.
12. Schwarz, L.B. Multi-Level Production/Inventory Control Systems: Theory and Practice. North-Holland, Amsterdam (1981)
13. Schwarz, L.B. Physical distribution: The anaylsis of inventory and location. AIIE Transactions **13** (1981) 138–150.
14. Schwarz, L.B., Schrage, L. On echelon holding costs. Management Science **24** (1978) 865–866.
15. Silver, E.A., Pyke, D.F., Peterson, R. Inventory Management and Production Planning and Scheduling. John-Wiley and Sons, New York (1998)
16. Wagner, H.M., Whitin, T.M. Dynamic version of the economic lot size model. Management Science **5** (1958) 89–96.
17. Zangwill, W. I. A backlogging model and a multi-echelon model of a dynamic economic lot size production system – a network approach. Management Science **15**(9) (1969) 506–527.

Table 6. Hybrid Results (secs/nodes): "–" indicates no solution within 1 hour

	Conventional Model							
	IC 1–6				IC 1–5			
	*H*1	*H*2	*H*3	*H*4	*H*1	*H*2	*H*3	*H*4
A1	347/41492	333/39447	281/34176	326/38977	994/143328	558/74616	377/45332	1650/277548
A2	644/88034	733/104322	613/80327	717/95789	736/136099	341/56385	717/130762	757/144476
A3	1030/144091	1360/201738	966/138839	1290/220931	1510/234798	1430/211116	639/74665	1390/208323
A4	137/18844	132/17806	117/14414	495/106910	308/67551	423/74196	313/56357	473/116441
A5	103/13737	155/23868	116/22690	103/14000	1260/172893	1430/206921	681/86815	1890/306806
A6	1250/176606	1150/158708	1160/182510	1510/236411	834/153700	1490/300912	810/150465	1340/278094
A7	160/24777	196/30297	177/27404	79.1/8760	287/65566	471/93533	195/32843	319/66657
A8	2770/509319	3180/608744	1600/283283	2320/428228	2010/318452	–/	1270/187083	2430/432682
A9	68.2/7312	93.2/11386	57.7/5955	68.7/7575	321/52203	205/31931	445/76195	291/45573
A10	754/106548	333/41473	820/118148	1460/159283	504/59214	565/82829	396/41762	411/49158
S1	3.81/282	3.63/298	4.12/366	4.65/468	2.85/309	2.59/244	3.02/306	4.23/505
S2	4.34/156	4.16/138	4.36/151	4.32/152	3.63/158	3.61/154	4.03/161	3.61/157
S3	11.1/1825	10.3/1650	10.8/1965	11.8/2092	7.57/1171	7.71/1013	6.13/789	9.09/1654
S4	12.9/2387	13.8/2445	15.1/2780	12.9/2327	14.1/2317	9.68/1322	15.3/2607	14.2/2326
S5	29.4/2310	39.1/2083	28.5/2173	43/3468	53.2/5134	46.5/4182	51.9/5146	44.5/3844
S6	5.19/328	4.72/278	5.19/291	6.51/490	3.88/295	3.59/267	4.14/282	5.51/509
S7	7.99/877	8.39/946	7.97/922	7.96/894	9.51/1160	9.31/1133	9.76/1141	8.81/1107
S8	31.1/3960	46.2/5028	44.1/3895	49.4/3989	60.5/4420	47.2/4186	50.7/4393	63.7/5082
S9	22.1/1930	19.7/1520	23.2/1814	25.1/1994	33.4/3326	34.1/3190	34.1/2955	27.3/2820
S10	6.72/1045	6.39/876	6.41/952	8.26/1429	5.52/998	5.78/994	5.88/980	6.72/1332
S11	4.52/280	4.12/248	5.01/331	4.34/273	4.27/268	4.33/271	5.01/360	4.41/269
S12	9.03/692	8.19/708	9.51/929	8.43/723	10.1/903	9.58/872	9.48/915	8.96/862
S13	4.27/893	4.46/882	4.26/880	4.27/903	3.28/807	3.52/825	3.71/963	3.33/855
S14	7.99/769	8.12/717	8.28/782	7.92/790	7.11/796	7.21/755	7.54/793	7.34/806
S15	8.5/1132	8.11/1065	8.37/1076	8.73/1121	6.91/926	6.65/886	7.02/973	6.64/911
W1					429/89478	438/88329	380/74273	558/129634
W2					908/187510	230/39146	963/203803	1010/226320
W3					41.1/6720	41.5/6949	40.3/6730	27.1/4098
W4					1810/425816	2140/542637	977/217213	1750/439325
W5					280/62574	136/27827	216/44857	143/28179
W6					64.8/13384	73.4/16783	74.2/17189	258/76382
W7					1130/284513	372/73895	971/239308	359/68076
W8					12.4/1664	11.6/1587	14.6/2046	14.8/2169
W9					37.6/8751	41.4/9720	37.4/8674	38.6/8999
W10					163/37997	105/21172	164/37934	108/22394

	Echelon Model							
	IC 1–6				IC 1–5			
	*H*1	*H*2	*H*3	*H*4	*H*1	*H*2	*H*3	*H*4
A1	422/50788	479/58022	292/34001	391/46499	1020/116263	408/42720	1070/131736	913/102886
A2	535/61891	551/68849	481/54697	299/29663	911/158986	581/96083	1190/231450	896/159848
A3	1630/203695	3000/511423	947/112375	1570/212956	1360/186429	1490/215757	1930/284210	1800/286333
A4	321/58663	262/41848	391/72124	243/38840	929/126296	988/132078	545/68024	1090/161570
A5	699/97698	453/61100	705/100947	575/79481	1430/193749	979/129565	822/105536	1900/277802
A6	1430/152051	1500/157195	2350/277142	1450/152150	1480/249527	2380/434608	1100/186745	1930/350496
A7	195/26560	230/33858	175/24016	230/36546	567/79609	496/65115	569/78103	1540/374719
A8	2090/318374	3160/507131	1670/223513	2130/342494	–/	–/	–/	–/
A9	78.7/8845	86.1/9279	148/20924	173/24650	286/42423	325/49181	386/73691	733/129922
A10	446/38123	551/48872	312/26625	657/99601	925/97216	1090/122752	454/49350	1130/144345
S1	3.67/310	3.15/231	3.65/308	3.41/300	3.32/404	3.65/391	3.67/449	5.71/826
S2	4.14/179	4.78/214	4.28/181	4.04/174	3.33/168	3.97/213	3.43/175	3.35/168
S3	12.3/1786	12.3/1753	14.1/1965	14.6/2252	10.4/1663	13.4/2220	11.3/1789	10.4/1689
S4	17.2/2094	17.4/1993	16.1/1791	19.1/2288	13.5/1680	13.5/1823	15.4/1777	13.4/1718
S5	36.9/2714	35.6/2633	38.3/3128	40.7/3580	21.2/1742	21.3/1726	22.5/1777	22.1/1742
S6	4.51/311	4.29/250	4.54/305	4.26/304	3.77/257	3.09/196	5.15/387	3.63/255
S7	6.79/735	6.02/503	6.53/583	6.52/611	9.35/1184	10.1/1387	8.81/1115	8.29/1026
S8	89.5/6930	68.6/5892	89.1/6846	80.3/6879	95.9/8350	108/8886	103/8718	106/9129
S9	19.6/1304	11.8/824	17.5/1257	15.3/1471	9.56/1022	7.84/648	9.78/953	11.5/1293
S10	7.41/975	7.52/960	6.88/885	7.51/1246	6.16/895	6.18/878	5.61/834	6.56/1212
S11	4.05/226	4.24/251	5.11/349	3.89/232	3.34/237	3.44/234	4.12/316	3.36/234
S12	7.21/640	7.13/595	8.42/864	7.34/630	6.55/812	6.56/750	6.38/812	6.48/727
S13	4.86/798	5.14/779	4.87/830	4.78/777	4.26/818	4.38/802	4.71/899	4.12/820
S14	6.84/719	8.88/622	5.85/507	7.05/728	7.34/804	7.36/837	6.87/793	7.18/797
S15	8.37/971	7.92/852	8.52/1094	8.21/891	4.46/728	4.68/707	4.51/695	4.61/698
W1					843/203927	649/144827	887/204364	1680/425469
W2					323/67383	386/79550	154/30806	508/103409
W3					29.9/4406	75.4/14194	29.5/4318	33.1/4902
W4					1960/469245	1870/452318	976/216520	2080/502255
W5					329/68894	113/21084	337/71589	200/38225
W6					80.6/15167	83.7/15878	86.5/16902	91.2/18139
W7					718/149184	489/92839	416/73521	253/36753
W8					16.4/2358	20.1/2943	19.3/2783	27.9/4977
W9					45.1/10149	63.3/15835	55.1/13012	42.8/9549
W10					54.1/11829	90.8/23117	56.5/13337	61.7/14019